UDK 517

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On some numerical results for the solution of a Stokes-Brinkman boundary-value problem

Abstract: *We study a model problem on the filtration of a conducting fluid through a porous layer. The filtrating flow is governed by the Stokes-Brinkman system which takes into the account the electric properties of the fluid. We do numerical simulations to obtain some specific properties on the considered Stokes-Brinkman system.*

Key words*: fluid flow, Stokes-Brinkman equation, numerical validation*

The current research is devoted to a problem of a liquid filtration through a porous layer. Such a process can be described by the Stokes–Brinkman boundary-value problem for flow of a viscous incompressible fluid in a porous medium and in transition regions where both viscous and filtration effects must be taken into account. The numerical solution of this problem is of interest for filtration theory, membrane technologies, biomechanics, and geophysics. Due to the complexity of analytical solution in practically important geometries, numerical approaches have become the main line of development. There are researchers and research groups that contributed to the analysis of the Stokes–Brinkman problem. We refer to the research school by professor A.N. Filippov (see e.g.[1-5]).

The topic of filtration process was investigated in literature mostly from physical and chemical point of view. The filtration through a spherical cell was investigated in [9]. The physical problem was stated formally mathematically in [9] and the existence of the weak solution was proved. The analytical solution for the considered system was derived and analyzed for a particular case of Brinkman's viscosity which illustrates the presence of superhydrophobic surfaces in a porous system.

In present paper, we investigate a more general problem on analysis of flow characteristics in the case when the thickness of the electric double layers is not neglected. This approach leads to analysis of the Poisson-Boltzmann equation in conjunction with the Stokes equations for a

conducting viscous incompressible liquid (electrolyte solution) and the Brinkman equations for a porous medium. The mentioned system of equations can not be solved analytically in general. However, it was possible to do some numerical treatment for the liquid velocity, ion flux density, electric potential, and concentration, depending on the ratio between the dimensionless thickness of the electric double layer, the Peclet number.

Coming to the rigorous statement of the problem, we consider a porous cell $\Omega = \Omega^i \cup \Omega^o \in R^3$ which is a spherical particle:

$$\Omega^i = \{0 < r < a, \varphi \in [0,2\pi], \theta \in [-\pi, \pi]\}, \Omega^o = \{a < r < b, \varphi \in [0,2\pi], \theta \in [-\pi, \pi]\}.$$

Its boundary is denoted by $\partial\Omega = \Gamma^i \cup \Gamma^o$, where

$$\Gamma^i = \{r = a\}, \Gamma^o = \{r = b\}.$$

Flow of fluid (electrolyte) in the outer domain Ω^o can be described by Stokes equation under low Reynold's number which involves also electromass force:

$$\nabla p^o = \mu^o \Delta v^o - \rho^o \nabla \varphi^o,$$

where

$$\rho^o = F_0$$

is the volumetric density of movable electric charges in a porous particle, $Z_{\pm,}Z_-$ are charge modules of cations and anions of the electrolyte, $C_{\pm}^o, C_-{}^o$ are concentrations of cations and anions; F_0 is Faraday constant. We assume that the liquid is incompressible:

$$\nabla \cdot v^o = 0.$$

Here p^o is local pressure, v^o is velocity vector, μ^o is a dynamic viscosity, φ^o is the electric potential.

Fluid flow is subjected to the Brinkman's equation with mass electric force in the inner domain Ω^i:

$$\nabla p^i = \mu^i \Delta v^i - \rho^i \nabla \varphi^i - k v^i,$$

where

$$\rho^i = F_0$$

is the volumetric density of movable electric charges in a porous particle, k is the Brinkman's constant which is inversely proportional to the particle's permeability. Here μ^i is the coefficient of Brinkman's viscosity. Brinkman's liquid is assumed to be incompressible as well:

$$\nabla \cdot v^i = 0.$$

Electric potential satisfies to Poisson's equation. The charge conservation equations must be valid both in the inner and outer layers

$$\nabla \cdot J_{\pm}^i = 0, \nabla \cdot J_{\pm}^o = 0.$$

. Moreover, the Nernst equation for densities $J_{\pm}^i$ of ion's fluxes in the porous particle holds.

$$J_{\pm}^{i} = v^{i} C_{\pm}^{i} - D_{m\pm}\left(\nabla C_{\pm}^{i} \pm Z_{\pm} C_{\pm}^{i} \nabla\, \varphi^{\,i} \frac{F_0}{RT}\right).$$

Here $D_{m\pm}$ are diffusion coefficients of electrolyte ions inside the porous particle.

Thus, the Stokes-Brinkman system describing the filtration process reads as follows in the dimensionless variables:

The following notations for dimentionless variables were used:

$$m_{\kappa} = \begin{cases} 1, \text{ if } \kappa = o, \\ m, \text{ if } \kappa = i, \end{cases} \mu_{\kappa} = \begin{cases} 0, \text{ if } \kappa = o, \\ ms^2, \text{ if } \kappa = i, \end{cases}$$

$$\sigma^{\kappa} = \begin{cases} 0, \text{ if } \kappa = o, \\ \sigma, \text{ if } \kappa = i, \end{cases} \nu^{\kappa} = \begin{cases} \nu_{\pm}, \text{ if } \kappa = o, \\ \nu_{m\pm}, \text{ if } \kappa = i. \end{cases}$$

$$j_{\pm} = \frac{J_{\pm}}{U_0 C_0}, \nabla = a\nabla, \Delta = a^2 \Delta, \nu_{\pm} = \frac{D_0}{D_{\pm}}, \nu_{m\pm} = \frac{D_0}{D_{m\pm}}, \mathrm{Pe} = \frac{aU_0}{D_0}, \sigma = \frac{\rho_V}{F_0 C_0}$$

$$p_0 = RTC_0, U_0 = \frac{ap_0}{\mu^o}, \delta = \frac{d}{a}, m = \frac{\mu^i}{\mu^o}, s^2 = \frac{a^2 k}{\mu^i}, s_0^2 = ms^2 = \frac{a^2}{R_b^2},$$

where $R_b = \sqrt{\frac{\mu^o}{k}}$ is the Brinkman's radius (characteristic thickness of the filtration layer), $C_0 = C_{e\pm} Z_{\pm}$ is the equivalent concentration of electrolyte equilibrated with a membrane, D_0 is diffusion coefficient scale, $d = \left(\frac{C_0 F_0^2}{\varepsilon \varepsilon_0 RT}\right)^{\frac{-1}{2}}$ is the Debye radius.

On the border $r = 1$ the following assumptions are assumed:

- Continuity of velocity field: $v^o = v^i$;
- Continuity of stress tensor: $\Sigma^o = \Sigma^i$;
- Continuity of the radial component of ion's fluxes: $j_{\pm}^{\mathrm{o}} \cdot n = j_{\pm}^{\mathrm{i}} \cdot n.$
- Continuity of the electric potential φ and both components of the electric field strength vector: $\varphi^o = \varphi^i, \nabla\varphi^o = \nabla\varphi^i$;

On the outer boundary $r = \frac{1}{\gamma}$ it is required:

- Cunningham conditions on the velocity: $v^o = U$;
- Zero gradient of ion concentrations:

$$C_{\pm}^{\mathrm{o}}\big|_{r=\frac{1}{\gamma},\theta=0} = C_{\pm}^{\mathrm{o}}\big|_{r=\frac{1}{\gamma},\theta=\pi}.$$

- Zero gradient of electric potential:

$$\varphi^{\mathrm{o}}\big|_{r=\frac{1}{\gamma},\theta=0} = \varphi^{\mathrm{o}}\big|_{r=\frac{1}{\gamma},\theta=\pi}$$

The numerics present the following Lebesgue space norm distribution depending on the Peclet number Pe (see Fig. 1) which characterizes the ratio between diffusion and the advection behavior in the fluid.

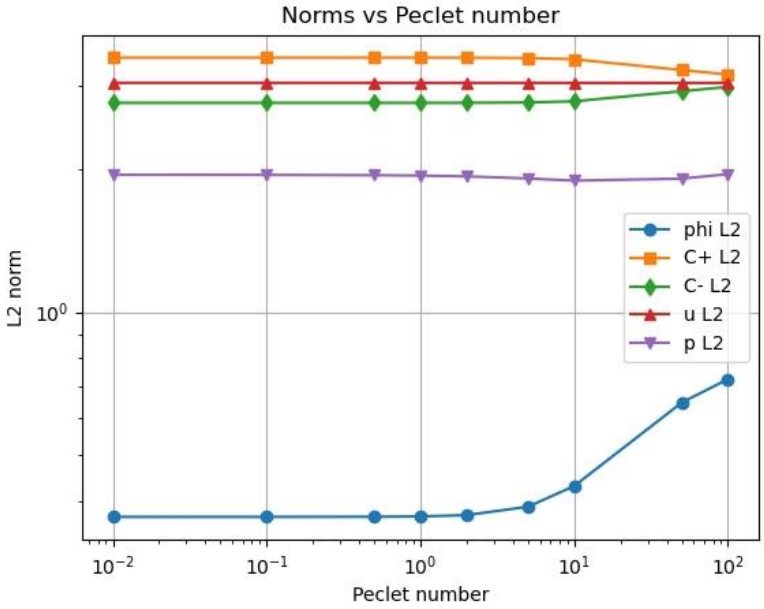


Fig. 1

The comparison of the system behavior depending on the Debye radius variation one can see on Fig.2.

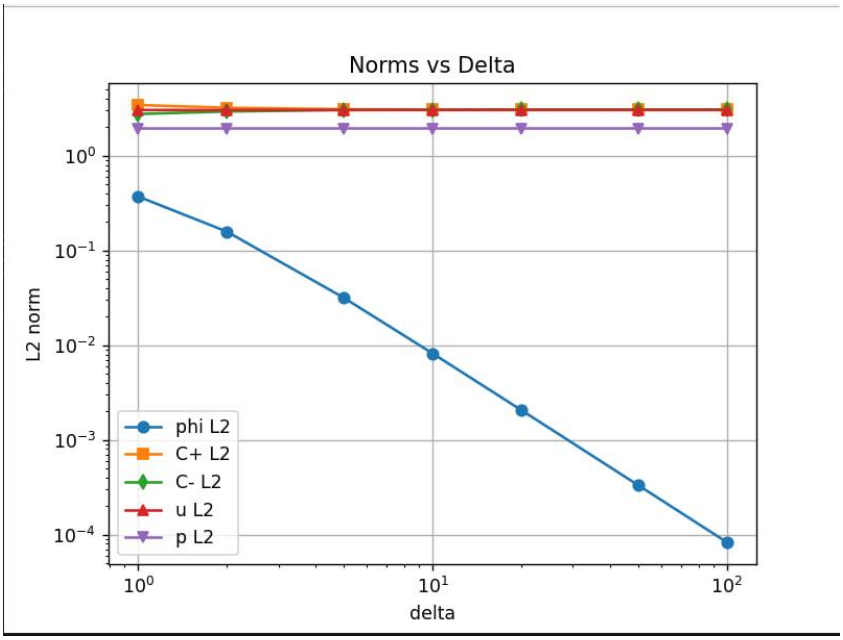


Fig. 2

The variation of Debye radius parameter almost does not effect on the velocity, pressure and concentrations of the flow. However, the electric potential decreases with respect to δ.

The graphics of the unknown solutions are presented below at Fig. 3-.4 as functions of particle's radius.

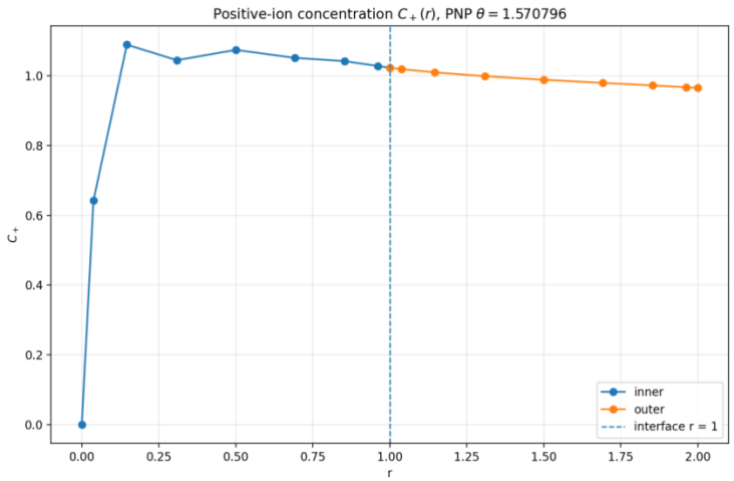


Fig. 3a

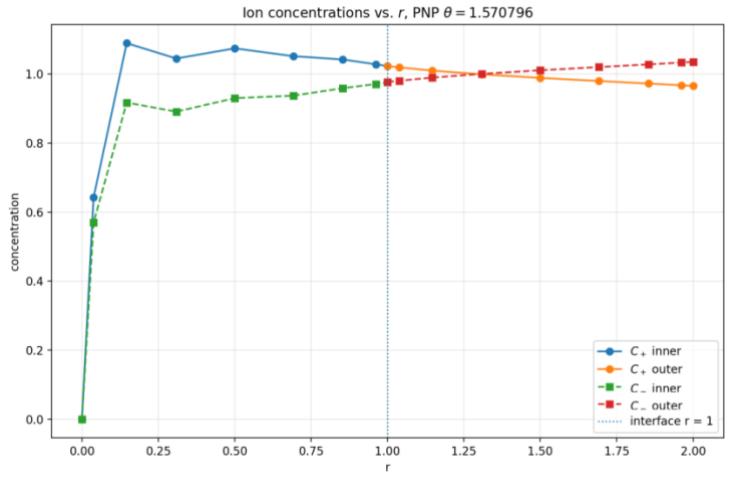


Fig. 3b

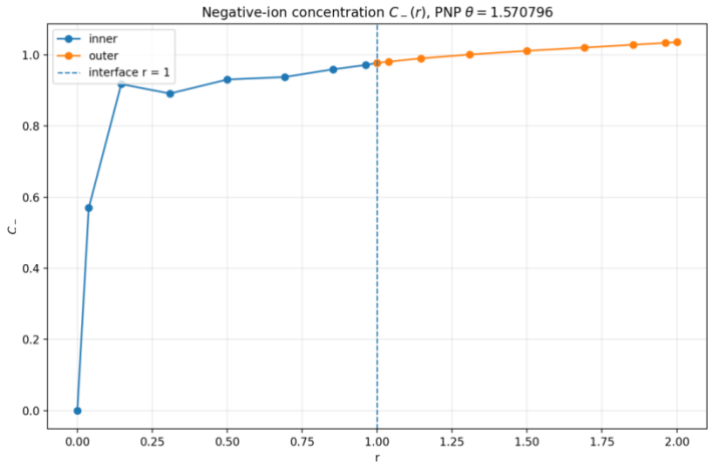


Fig. 3c

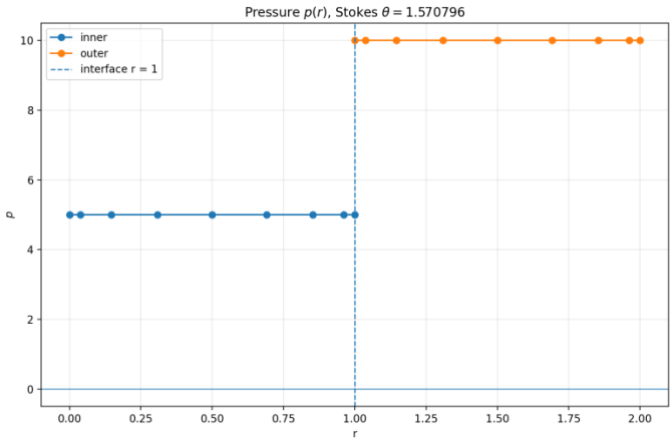


Fig. 3d

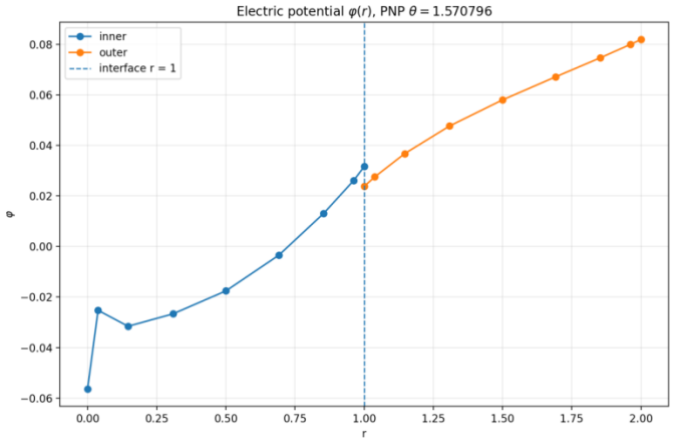


Fig. 3e

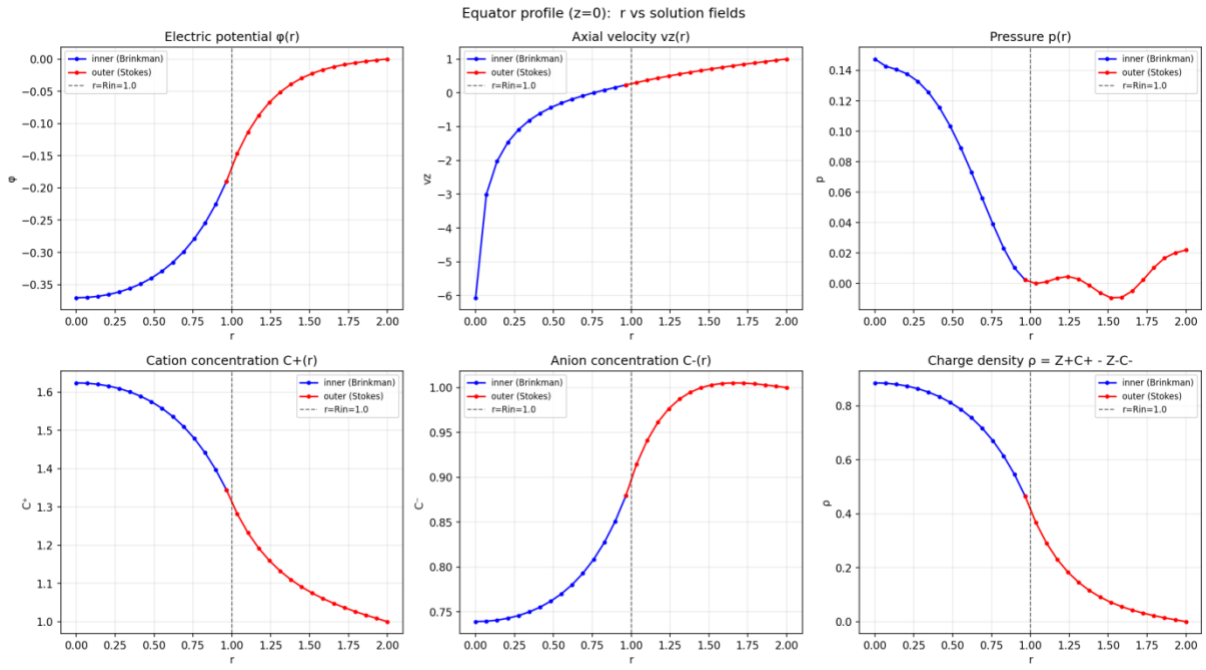


Fig. 4

In our research we have considered the dependence on the viscosity as well. Based on the numerical simulations, one can conclude that the pressure behavior was a monotonically increasing dependence on the fluid viscosity.

The velocity and electric potential of fluid are an increasing functions with respect to radius and it behaves similarly for all cases of the viscosity coefficient. The cation concentration is a monotonically decreasing function of radius while the anion concentration function increases on its domain independently on the viscosity m. These remarks are agrees with theoretical estimates derived in paper [9].

The fact on the ion concentration can be explained by double electric layer near the charged surface. The Brinkman's boundary area is governed by both viscous friction of a fluid and resistance of porous medium. Therefore electrostatic forces play the crucial role for the charged particle's distribution.

Our analysis proves that the wider the outer layer with Stokes flow, the greater value can take the permeability of the considered system. The greater the Debye radius the greater values takes the permeability coefficient.

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